PDE I MTH 847 QUALIFYING EXAM January 2, 2020

Name:__________________________ Signature:__________________________

Write clearly and coherently! Show all your work! This is a closed book exam, notes, electronic devices, cell phones, etc., can not be used. You can use a 4×6 index card. Read all problems through once before beginning your work. Problem 7 is optional for extra credit.

Problem	Points
1.	
2.	
3.	
4.	
5.	
6.	
7.	
Total:	

Problem 1. [10 points] Find the solution of the following initial value problem for the transport equation

$$u_t + b \cdot Du + cu = f \ \text{ in } \mathbf{R}^n \times (0, \infty), \ u = g \ \text{ on } \mathbf{R}^n \times \{t = 0\}.$$

Here $c \in \mathbf{R}$, $b \in \mathbf{R}^n$ are constants, $f(x,t) = e^t$, $g(x) = |x|^2$.

Problem 2. [10 points] Let $\mathbf{B}(0,1) = \{x \in \mathbf{R}^2 : \ |x| < 1\}$ be the open unit ball.

(a) Find the solution of the boundary value problem

$$\begin{cases} \Delta u &= 0 & \text{in } \mathbf{B}(0,1) \\ u(x) &= 4x_1^2 - 4x_1 + 2x_2^2 & \text{on } \partial\mathbf{B}(0,1). \end{cases}.$$

(b) Find the value $u(0)$.

(c) Find $\max\{u(x), \ x \in \overline{\mathbf{B}(0,1)}\}$ and $\min\{u(x), \ x \in \overline{\mathbf{B}(0,1)}\}$.

(d) Is there a point $x \in \mathbf{B}(0,1)$ such that $u(x) = 0$?

Problem 3. [10 points] (a) Let a be a number, $0 < a < 1$. Consider the set

$$A = \{(x,t) \in \mathbf{R}^n \times (0,\infty) : \ |x| \geq 1, \ 0 < t < a\}.$$

Prove that the wave equation

$$u_{tt} - \Delta u = 0, \quad \text{in } \mathbf{R}^n \times (0,\infty)$$

has a solution $u \in C^2(\mathbf{R}^n \times (0,\infty))$ that is not identically zero, but vanishes on the set A.

(b) Consider the initial value problem for the $n = 3$ dimensional wave equation

$$u_{tt} - \Delta u = 0, \quad \text{in } \mathbf{R}^3 \times (0,\infty), \quad u = 0, \ u_t = h \text{ on } \mathbf{R}^3 \times \{t = 0\}.$$

The function $h \in C_c^\infty(\mathbf{R}^3)$ satisfies the conditions that $0 \leq h(x) \leq 1$ for all $x \in \mathbf{R}^3$, $h(x) = 1$ if $|x| \leq 1$ and $h(x) = 0$ if $|x| \geq 2$. Calculate the values of $u(x,t)$ at the points $(0,0,0,\frac{1}{2})$, $(0,0,0,3)$, $(6,0,0,4)$, and the value of $u_t(x,t)$ at $(0,0,0,\frac{1}{2})$.

Problem 4. [10 points] Solve the mixed initial-boundary value problem for the wave equation in one dimension

$$\begin{cases} u_{tt} = u_{xx}, & x > 0,\ t > 0, \\ u(x,0) = g(x),\ u_t(x,0) = 0, & x > 0, \\ u_t(0,t) = 4u_x(0,t) & t > 0, \end{cases}$$

where the function $g \in C^2(\mathbf{R}_+)$ vanishes near $x = 0$.

Problem 5. [10 points] Let U be a bounded open set in $\mathbf{R}^n$ with C^1 boundary. The time $T > 0$ is given Let $U_T = U \times (0, T]$, and $\Gamma_T = \overline{U_T} - U_T$. Let $q \in C(U)$ be a nonnegative function and $f \in C(U_T)$, $g \in C(\Gamma_T)$. Consider the boundary value problem

$$\begin{cases} u_t - \Delta u + q(x)u &= f \quad \text{in } U_T \\ u &= g \quad \text{on } \Gamma_T. \end{cases}$$

Prove that there is at most one solution $u \in C^{2,1}(\overline{U_T})$ of this boundary value problem.

Problem 6. [10 points] Let $f \in C(\mathbf{R}^n \times [0,\infty))$ and $g \in C(\mathbf{R}^n)$ be bounded functions.

(1) Prove that there is a solution of the nonhomogeneous heat equation

$$u_t - \Delta u = f \text{ in } \mathbf{R}^n \times (0,\infty), \quad u(x,0) = g(x) \text{ on } \mathbf{R}^n$$

satisfying the estimate

$$|u(x,t)| \leq \sup\{|g(y)| : y \in \mathbf{R}^n\} + t \sup\{|f(y,s)| : y \in \mathbf{R}^n,\ 0 \leq s \leq t\}$$

using Duhamel's principle.

(2) Prove that there is a bounded solution u if $|f(x,t)| \leq \frac{1}{1+t^2}$ for all $(x,t) \in \mathbf{R}^n \times [0,\infty)$.

Problem 7. [10 points] **Problem 7 is optional for extra credit**.

Use the method of characteristics to solve the first order equation

$$(u_{x_1})^2 + 2x_2^2 u_{x_2} = 1, \quad (x_1, x_2) \in \mathbf{R}^2$$

with the condition

$$u(x_1, 1) = \frac{x_1^2}{2},\ x_1 \in \mathbf{R}.$$